MASSIVE ELECTRODYNAMICS AND THE MAGNETIC MONOPOLES

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Abstract

We investigate in detail the problem of constructing magnetic monopole solutions within the finite-range electrodynamics (i.e., electrodynamics with non-zero photon mass, which is the simplest extension of the standard theory; it is fully compatible with the experiment). We first analyze the classical electrodynamics with the additional terms describing the photon mass and the magnetic charge; then we look for a solution analogous to the Dirac monopole solution. Next, we plug the found solution into the Schrödinger equation describing the interaction between the the magnetic charge and the electron. After that, we try to derive the Dirac quantization condition for our case. Since gauge invariance is lost in massive electrodynamics, we use the method of angular momentum algebra. Under rather general assumptions we prove the theorem that the construction of such an algebra is not possible and therefore the quantization condition cannot be derived. This points to the conclusion that the Dirac monopole and the finite photon mass cannot coexist within one and the same theory. Some physical consequences of this

conclusion are considered. The case of t'Hooft-Polyakov monopole is touched

upon briefly.

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I. INTRODUCTION

Massive electrodynamics is electrodynamics in which the photon has a small mass[1] rather than being exactly massless. It is perhaps the simplest and the most straightforward extension of the standard quantum electrodynamics (QED) [1,2]. It can be embedded into the standard $SU(2) \times U(1)$ model [3].

Although the introduction of a small photon mass may look unaesthetical, in some respects massive QED is simpler theoretically than the standard theory. For example, massive QED can be quantized in a manifestly Lorentz-covariant way without introducing the indefinite metric while the standard QED cannot. Also, the analysis of infrared properties of massive QED is easier because there are no infrared singularities caused by the zero photon mass.

Note that although the photon is massive in the theory under consideration, the electric charge is strictly conserved in massive QED as well as in standard QED. From the experimenatal point of view, massive QED is as perfect as standard QED.

In massive QED the photon has three polarization states: two transverse ones and one longitudinal. Despite this fact, it has been shown that the limiting transition between both theories is in fact smooth rather than discontinuous [4,5] . The physical reason is that the interaction of longitudinal photons gets weaker as the photon mass tends to zero so that in the massless limit they effectively decouple.

Such smoothness may even be elevated to a role of an important theoretical principle: each physical phenomenon which can be described in standard QED must have its counterpart description in massive QED; the two descriptions must merge continuously in the limit of vanishing photon mass.

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[1]Experimentally, the photon mass has to be very small: less than 10^{-24} GeV or even 10^{-36} GeV, but the existence of such a bound is not important for the purposes of this paper, so we do not go into detail here.

Independently of whether this conjecture is true or not, it is instructive to see how it works (or fails) in various physical contexts since in this way a better understanding of the physical situation can arise.

With this aspect in view, it is natural to consider those phenomena, for which the gauge invariance of QED is crucial or at least essential. One example is the Dirac monopole [6,7]. Although there have been much work devoted to various sides of magnetic monopole physics, the aspect we are interested in has not been analyzed in detail so far.

The purpose of the present paper is to try and fill that gap. In other words, we would like to know if magnetic monopoles and massive photons can coexist within one and the same theory.

We know that the very existence of Dirac monopole is tightly connected with the existence of gauge invariance of QED. In massive QED, there is no gauge invariance anymore. So, the question is: what happens to magnetic monopoles there? Do they survive the loss of gauge invariance?

Before starting any reasoning, we could just try to guess the right answer. Obviously, it has to be either "yes" or "no". Let us consider these in turn.

At first sight, the positive answer does not look impossible at all. We know that with the introduction of photon mass the electrostatic field changes from the usual Coulomb form, $E \sim \frac{1}{r^2}$, to the Yukawa form $E \sim \frac{1}{r^2}e^{-mr}$. So, we would expect the magnetic field of a monopole also to change from $H \sim \frac{1}{r^2}$ to $H \sim \frac{1}{r^2}e^{-mr}$. However, what would happen to the Dirac quantization cony of Melbourne, the magnitude of the photon mass and vanishing smoothly with the vanishing photon mass? Or should the quantization condition remain the same in both theories? Next, how can we make sure the string is unobservable if we do not have gauge invariance?

Now, if we guess that there can be no monopoles in massive QED, then again there arise several questions. What about rotational invariance? We know that the Dirac quantization condition can be obtained from rotational invariance and angular momentum quantization [8–11] without using gauge invariance. Can we generalize that kind of arguments to the

case of massive QED? Next, what about the limit of vanishing photon mass? If magnetic monopoles abruptly disappear when photon mass equals zero, then, how can we make a continuous transition between the massive and massless QED?

So it seems that both "yes" and "no" options offer interesting questions to ponder on.

Our startegy in this paper will be as follows. First we write down the classical Maxwell equations describing the magnetic charge and massive photon. We then find the solutions which would go into the Dirac monopole solution in the limit of vanishing photon mass (Section II). At this stage no inconsistency arise , but no genuine monopoles either, because we have of course the string attached to the monopole.

In the spirit of Dirac's approach we then turn to quantum mechanics in our efforts to eliminate the mischievous string. We undertake this step in Section III by plugging the solution found in Section II into the Shrödinger equation describing the interaction between the the magnetic charge and the electron. After that, we try to derive the Dirac quantization condition for our case. Since gauge invariance is lost in massive electrodynamics, we use the method of angular momentum algebra [8], generalized to the case of non-vanishing photon mass. Under rather general assumptions we prove a theorem that the construction of such an algebra is not possible and therefore the quantization condition cannot be derived. This points to the conclusion that the Dirac monopole and the finite photon mass cannot coexist within one and the same theory. Some physical consequences of this conclusion, including the problem of continuity, are discussed at a qualitative level in Section IV. The case of t'Hooft-Polyakov monopole is touched upon briefly at the end of the paper.

II. CLASSICAL THEORY

We start by writing down the Proca equation describing electrodynamics with finite range (or equivalently with a non-zero photon mass)[2]:

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[2]We use the Heaviside system of units throughout this paper and also put $\hbar = 1, c = 1$.

$$\partial^\mu F_{\mu\nu} = J_\mu - m^2 A_\nu \tag{1a}$$

$$\partial^\mu \tilde{F}_{\mu\nu} = 0 \tag{1b}$$

$$\partial^\mu A_\mu = 0, \tag{1c}$$

where, the field strength $F_{\mu\nu}$ is connected with the 4-vector potential A_μ the usual way:

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu.$$

The dual pseudo–tensor $\tilde{F}_{\mu\nu}$ is defined as usual via

$$\tilde{F}_{\mu\nu} = \frac{1}{2}\varepsilon_{\mu\nu\alpha\beta}F^{\alpha\beta};$$

J_ν is the (electric) current density. Next, the Maxwell equations generalized to include magnetic charge are:

$$\partial^\mu F_{\mu\nu} = J_\nu \tag{2a}$$

$$\partial^\mu \tilde{F}_{\mu\nu} = J_\nu^g \tag{2b}$$

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu, \tag{2c}$$

where index g denotes the magnetic current density.

Now the straightforward generalization of systems (12) and (2) reads

$$\partial^\mu F_{\mu\nu} = J_\nu - m^2 A_\nu \tag{3a}$$

$$\partial^\mu \tilde{F}_{\mu\nu} = J_\nu^g \tag{3b}$$

$$\partial^\mu A_\mu = 0 \tag{3c}$$

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu. \tag{3d}$$

Let us rewrite this system in the 3–dimensional form:

$$\nabla \cdot \mathbf{E} = \rho - m^2 A_o \tag{4a}$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{H}}{\partial t} - \mathbf{j}_g \tag{4b}$$

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$$\nabla \cdot \mathbf{H} = \rho_g \tag{4c}$$

$$\nabla \times \mathbf{H} = \mathbf{j} - m^2 \mathbf{A} + \frac{\partial \mathbf{E}}{\partial t} \tag{4d}$$

$$\frac{\partial A_o}{\partial t} + \nabla \cdot \mathbf{A} = 0 \tag{4e}$$

$$\mathbf{E} = -grad A_o + \frac{\partial \mathbf{A}}{\partial t} \tag{4f}$$

$$\mathbf{H} = \nabla \times \mathbf{A}. \tag{4g}$$

This is the system of generalized Maxwell equations which would presumably describe the existence of both magnetic charge and non-zero photon mass.

A few remarks are now in order.

1. The photon mass term $m^2 A_\mu$ in the right-hand side of Maxwell equations violates the symmetry between the electric and magnetic charges. This is clearly seen from the comparison of Eq. (4a) and (4c): the "electric" equation (4a has as its solution the familiar Yukawa potential

$$\mathbf{E} = \frac{q}{4\pi r^3} e^{-mr} \mathbf{r},$$

   while the "magnetic" equation (4c) does not feel the photon mass at all.

2. The gauge invariance is completely lost due to the photon mass term. Indeed, it can be seen that the transformation

$$A_\mu \to A_\mu + \partial_\mu f$$

   is inconsistent with equations (12) whatever the function f is. (Recall that the ordinary Maxwell equations in the Lorentz gauge, $\partial_\mu A^\mu = 0$, also do not allow gauge transformation with the arbitrary function f. However, these equations allow such transformations for f satisfying the condition $\Box f = 0$. In our case, even that restricted gauge invariance is lost.) Note that this loss of invariance has occurred already at the stage of the Proca equations without magnetic charge and so it has nothing to do with the introduction of magnetic charge.

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3. Due to the loss of gauge invariance, the vector potential A_μ becomes observable quantity on the same footing as the field strength $F_{\mu\nu}$. It can be seen that the presence of photon mass term $m^2\mathbf{A}$ in the righthand side of the equation of (4d) creates a sort of additional current density, in addition to the usual electric current $\mathbf{j}$.

4. It is not immediately obvious that the loss of gauge invariance destroys the consistency of the Dirac monopole theory and the validity of the quantization condition. For example, the Aharonov-Bohm effect which is also based on the electromagnetic gauge invariance, has been shown to survive in the massive electrodynamics despite the absence of gauge invariance there [12].

Our modified Maxwell equations tell us that there arises the additional magnetic field created by the "potential-current" $m^2\mathbf{A}$. There is no way to separate this additional magnetic field from the normal one. Although in Proca theory (without magnetic charge) this circumstance does not cause any problems, it becomes the main source of trouble once magnetic charges are added to the massive electrodynamics, as we shall see shortly.

After these general remarks, let us see if our system of "Maxwell + photon mass + magnetic charge" equations (4) is consistent or not. Let us try to find a static monopole-like solution of that system. For this purpose, we assume the absence of electric fields, charges and currents ($\mathbf{E} = 0$, $A_o = 0$, $\rho = 0$, $\mathbf{j} = 0$) as well as the absence of magnetic current ($\mathbf{j}_g = 0$).

We then are left essentially with four equations

$$\nabla \cdot \mathbf{H} = \rho_g \tag{5a}$$

$$\nabla \times \mathbf{H} = -m^2\mathbf{A} \tag{5b}$$

$$\mathbf{H} = \nabla \times \mathbf{A} \tag{5c}$$

$$\nabla \cdot \mathbf{A} = 0. \tag{5d}$$

The first equation has the familiar Dirac monopole solution:

$$\mathbf{H}^D = \frac{g}{4\pi r^3}\mathbf{r}, \tag{6a}$$

$$A_r^D = A_\theta^D = 0, \tag{6b}$$

$$A_\varphi^D = \frac{g}{4\pi r}\tan\frac{\theta}{2}. \tag{6c}$$

As is well known, this solution involves a singularity in vector potential along the line $\theta = \pi$ ("a string"). Yet this singularity was shown by Dirac to be only a nuisance without any physical significance. [3] Now if we plug this Dirac solution (or, more exactly, the Coulomb magnetic field) into the second equation, we immediately run into trouble, because clearly $\nabla \times \mathbf{H}^D = 0$, instead of being equal to $(-m^2\mathbf{A})$. Let us try to find a better solution by adding something to the Dirac solution. In this way we write:

$$\mathbf{H} = \mathbf{H}^D + \mathbf{H}', \quad \mathbf{A} = \mathbf{A}^D + \mathbf{A}', \tag{7}$$

where the rotor and the divergence of the additional field $\mathbf{H}'$ must satisfy

$$\nabla \cdot \mathbf{H}' = 0 \tag{8}$$

$$\nabla \times \mathbf{H}' = -m^2(\mathbf{A}^D + \mathbf{A}'), \tag{9}$$

while the divergence of the potential $\mathbf{A}'$ must vanish:

$$\nabla \cdot \mathbf{A}' = 0, \tag{10}$$

because

$$\nabla \cdot \mathbf{A}^D = 0 \text{ and } \nabla \cdot (\mathbf{A}^D + \mathbf{A}') = 0.$$

Finally,

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[3] The magnetic charge can be assumed to be either scalar or pseudoscalar under the action of P-parity (for more details see e.g. a review [13]). In the former case, which we adopt in this paper, the theory with magnetic monopoles is not parity invariant. However, none of our physical results would be changed if we treated the magnetic charge as a pseudoscalar (rather than scalar) quantity.

$$\nabla \times \mathbf{A}' = \nabla \times \mathbf{H}'. \tag{11}$$

Now, we have the complete system of equations (8) through (11) for the rotors and divergences of both $\mathbf{H}'$ and $\mathbf{A}'$.

Let us see if there is any solution to it. Taking the rotor of both sides of Eqn.(11) and using Eqns. (9) and (10) we get the second-order equation for $\mathbf{A}'$ only:

$$(\triangle - m^2)\mathbf{A}' = m^2\mathbf{A}^D. \tag{12}$$

The natural boundary condition for this equation is that $\mathbf{A}'$ must vanish at infinity. Note that after this equation is solved we have to make sure that the transversality condition $\nabla \cdot \mathbf{A}' = 0$ is obeyed.

In cartesian coordinates we get three decoupled scalar equations instead of one vector equation:

$$(\triangle - m^2)A_i' = m^2 A_i^D \ i = x, y, z. \tag{13}$$

The Green function for the equation

$$(\triangle - m^2)u = f,$$

with the boundary condition of vanishing at infinity is:

$$G(\mathbf{r}, \mathbf{r}') = \frac{1}{4\pi} \frac{exp(-m|\mathbf{r} - \mathbf{r}'|)}{|\mathbf{r} - \mathbf{r}'|}.$$

Therefore, we can write the solutions of the Eqns.(13) as

$$A_x'(\mathbf{r}) = \frac{m^2}{4\pi} \int d^3\mathbf{r}' A_x^D(\mathbf{r}') \frac{exp(-m|\mathbf{r} - \mathbf{r}'|)}{|\mathbf{r} - \mathbf{r}'|},$$

and the same for y, z components. Thus we can write in the vector form:

$$\mathbf{A}'(\mathbf{r}) = \frac{m^2}{4\pi} \int d^3\mathbf{r}' \mathbf{A}^D(\mathbf{r}') \frac{exp(-m|\mathbf{r} - \mathbf{r}'|)}{|\mathbf{r} - \mathbf{r}'|}. \tag{14}$$

Let us check that this solution is indeed transverse: find $\nabla \cdot \mathbf{A}'(\mathbf{r})$.

We have (here $\mathbf{R} = \mathbf{r} - \mathbf{r}'$):

$$\nabla \cdot f(R)\mathbf{g}(\mathbf{r}') = \nabla f(R) \cdot \mathbf{g}(\mathbf{r}') = f'(R) \cdot \nabla R \cdot \mathbf{g}(\mathbf{r}') = \frac{f'(R)}{R}\mathbf{R}\mathbf{g}(\mathbf{r}').$$

(Note that all differential operations are taken with respect to the vector $\mathbf{r}$.)

Next,

$$f'(R) = \frac{d}{dR}\left(\frac{e^{-mR}}{R}\right) = -\frac{e^{-mR}}{R^2}(1+mR).$$

Finally we obtain

$$\nabla \cdot \mathbf{A}'(\mathbf{r}) = -\frac{m^2}{4\pi}\int d^3\mathbf{r}'\frac{e^{-mR}}{R^3}(1+mR)\mathbf{R}\mathbf{A}_D(\mathbf{r}').$$

Let us now show that it is zero. Note that l.h.s (and r.h.s) are *pseudoscalar* (because both $\mathbf{A}'$ and $\mathbf{A}_D$ are *pseudovectors*). To construct this pseudoscalar, we have only two vectors at our disposal: $\mathbf{r}$ and $\mathbf{n}$ (the unit vector along the monopole string). From them, we can make only two pseudoscalars: $(\mathbf{r} \times \mathbf{n})\mathbf{n}$ and $(\mathbf{r} \times \mathbf{n})\mathbf{r}$. Both of then are zero (in addition to that, the first combination is ruled out by the condition that it must be linear in $\mathbf{n}$, but not quadratic).

Thus we have shown, that our solution satisfies both the equation $(\triangle - m^2)\mathbf{A}' = -m^2\mathbf{A}^D$ and the subsidary condition of transversality, $\nabla \cdot \mathbf{A}' = 0$.

Let us now find the restrictions on the form of the potential $\mathbf{A}'(\mathbf{r})$ which follow from the general principles (dimensional analysis, rotational invariance and space reflection).

The most general form of $\mathbf{A}'(\mathbf{r})$, as a *pseudovector* depending only on two vectors, $\mathbf{r}$ and $\mathbf{n}$, is:

$$\mathbf{A}'(\mathbf{r}) = \tilde{f}(r, \mathbf{n}\mathbf{r})(\mathbf{n} \times \mathbf{r}). \tag{15}$$

It is easier to work with dimensionless quantities, so let us write again our initial formula

$$\mathbf{A}'(\mathbf{r}) = \frac{m^2}{4\pi}\int d^3\mathbf{r}'\mathbf{A}^D(\mathbf{r}')\frac{e^{-mR}}{R},$$

and make a change of variables

$$\mathbf{r'}_1 = m\mathbf{r'}$$

$$\mathbf{r}_1 = m\mathbf{r}$$

$$\mathbf{R}_1 = m\mathbf{R},$$

so that $\mathbf{r'}_1$, $\mathbf{r}_1$, $\mathbf{R}_1$ are all dimensionless.

Now,

$$d^3\mathbf{r'} = \tfrac{1}{m^3}d^3\mathbf{r'}_1$$

$$\mathbf{A}^D(\mathbf{r'}) \sim \tfrac{1}{\mathbf{r'}} = \tfrac{m}{r'_1}$$

$$R = \tfrac{R_1}{m}.$$

So that we get:

$$\mathbf{A'}(\tfrac{\mathbf{r}_1}{m}) = \tfrac{m^2}{4\pi}\tfrac{1}{m}\int d^3\mathbf{r'}_1\tfrac{exp(-R_1)}{R_1}\left(-\tfrac{g}{4\pi r'_1}\tfrac{\mathbf{n}\times\mathbf{r'}_1}{r'_1-(\mathbf{nr'}_1)}\right)$$

$$= mgf(r_1,\mathbf{r}_1\mathbf{n})(\mathbf{n}\times\mathbf{r}_1).$$

Now, go back to the old variables:

$$\mathbf{A'}(\mathbf{r}) = m^2 gf(mr, m(\mathbf{rn}))(\mathbf{n}\times\mathbf{r}). \tag{16}$$

Now f is a dimensionless function of two variables.

Let us check if this most general form satisfies the condition of transversality, $\nabla\cdot\mathbf{A'} = 0$.

Use:

$$\nabla\cdot(S\mathbf{V}) = S\nabla\cdot\mathbf{V} + \mathbf{V}\nabla S$$

$$\Rightarrow \nabla\cdot\mathbf{A'} = m^2 g\{f\nabla\cdot(\mathbf{n}\times\mathbf{r}) + \mathbf{n}\times\mathbf{r}\nabla f\}.$$

But

$$\nabla\cdot(\mathbf{V}_1\times\mathbf{V}_2) = V_2\nabla\times\mathbf{V}_1 - \mathbf{V}_1\nabla\times\mathbf{V}_2.$$

Then,

$$\nabla\cdot(\mathbf{n}\times\mathbf{r}) = 0.$$

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Therefore

$$\nabla \cdot \mathbf{A}' = m^2 g(\mathbf{n} \times \mathbf{r})\nabla f.$$

Denote

$$f(mr, m(\mathbf{rn})) = f(x, y)|_{x=mr, y=m(\mathbf{rn})}.$$

Then,

$$\nabla f = \frac{\partial f}{\partial x} m \cdot \nabla r + \frac{\partial f}{\partial y} m \cdot \nabla(\mathbf{rn}).$$

But

$$\nabla r = \frac{\mathbf{r}}{r}, \ \nabla(\mathbf{rn}) = \mathbf{n}.$$

Therefore,

$$\nabla f = \frac{\partial f}{\partial x} \cdot m \cdot \frac{\mathbf{r}}{r} + \frac{\partial f}{\partial y} m \cdot \mathbf{n}.$$

Hence we see that both terms $\sim \mathbf{r}$ and $\sim \mathbf{n}$ give zero when multiplied by the vector product $(\mathbf{n} \times \mathbf{r})$.

Conclusion: our general form for $\mathbf{A}'$ *always* satisfies the transversality condition.

Now consider the problem of possible singularities in $\mathbf{A}'(\mathbf{r})$.

In principle, there are two potential sources of singularities within $\mathbf{A}'(\mathbf{r})$:

1. The $\frac{1}{R}$ behaviour of Green's function.

2. The singularity of string due to $\mathbf{A}^D$ factor.

The worst case is when they occur simultaneously. Let us consider this case. That is, we consider the case, when the observation point(O) lies on the string, while the integration point (I) approaches it (i.e. observation point).

To see what happens near the observation point, let us shift the integration variables:

$$\mathbf{r}' = \mathbf{r} - \mathbf{R}, \ d^3\mathbf{r}' = d^3\mathbf{R}.$$

Now choose the coordinates: see Fig.1. Put the origin of the spherical system of coordinates at the point M (i.e. where monopole is), z–axis directed opposite the string of monopole. Call this system M.

Now, introduce a second spherical system with the origin at O. Call this system O. Obviously, $d^3\mathbf{R}$ has a simple form in the system O:

$$d^3\mathbf{R} = R^2 dR \sin\theta_0 d\theta_0 d\phi,$$

while the vector potential has a simple form in the system M:

$$\mathbf{A}_D(I)_{I\to 0} = \frac{g}{4\pi r} \cdot tg\frac{\theta_M}{2} \approx \frac{g}{4\pi r}\frac{2}{\pi - \theta_M}\mathbf{e}(\phi),$$

where $\mathbf{e}(\phi)$ is a unit vector in the azimuthal direction.

Now, we need to find the relation between the angles θ_M and θ_0.

For this purpose consider $\triangle IOM$: in it, we know two sides, $OM = r$ and $OI = R$, and one angle, $\angle IOM = \theta_0$. If we find the angle, $\angle OMI = x$ then θ_M will be simply $\theta_M = \pi - x$. We have:

$$tg\, x = \frac{IH}{HM} = \frac{R\sin\theta_0}{r - R\cos\theta_0} \approx \frac{R}{r}\sin\theta_0.$$

Since x is assumed small, we obtain:

$$x \approx \frac{R}{r}\sin\theta_0.$$

But

$$\mathbf{A}'(\mathbf{r}) \approx \frac{g}{4\pi r}\frac{2}{x}\mathbf{e}(\phi) = \frac{g}{2\pi} \cdot \frac{1}{R\sin\theta_0} \cdot \mathbf{e}(\phi).$$

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Putting all together, we get:

$$\mathbf{A}'(\mathbf{r}) \approx \frac{g}{2\pi} \int R^2 dR \sin\theta_0 d\theta_0 d\phi \cdot \frac{1}{R} \cdot \frac{1}{R\sin\theta_0}\mathbf{e}(\phi)$$

$$\approx \frac{g}{2\pi} \int dR \int_0^{2\pi} d\phi\, \mathbf{e}(\phi).$$

From this form, it is clear that $\mathbf{A}'(\mathbf{r})$ does not have any singularity in the case considered. (Note also, that $\int_0^{2\pi} d\phi\, \mathbf{e}(\phi) = 0$.) But because we have considered the *worst* possible case, we may conclude that *there are no singularities in* $\mathbf{A}'(\mathbf{r})$ *at all*.

Let us now find the structural form of the magnetic field $\mathbf{H}'(\mathbf{r})$.

From Eqn.(14) which gives us the integral expression for the vector potential $\mathbf{A}'$, we can obtain the formula for $\mathbf{H}'$ by taking rotor:

$$\mathbf{H}'(\mathbf{r}) = \nabla \times \mathbf{A}'(\mathbf{r}) = \frac{m^2}{4\pi} \int d^3\mathbf{r}' \frac{e^{-mR}}{R^3}(1 + mR)(\mathbf{A}_D \times \mathbf{R}). \tag{17}$$

Now, because $\mathbf{H}'(\mathbf{r})$ is a vector (not a pseudovector!) depending on the two vectors only, $\mathbf{n}$ and $\mathbf{r}$, it has the following most general form:

$$\mathbf{H}'(\mathbf{r}) = h(r, \mathbf{nr})\mathbf{r} + g(r, \mathbf{nr})\mathbf{n}. \tag{18}$$

A question arises: can $\mathbf{H}'(\mathbf{r})$ be spherically symmetric, that is, can it take the form

$$\mathbf{H}'(\mathbf{r}) = h(r) \cdot \mathbf{r} \quad ?$$

The answer is no, because in that case we would have

$$\nabla \times \mathbf{H}'(\mathbf{r}) = 0$$

everywhere, which is inconsistent with the initial Maxwell equations.

III. QUANTUM MECHANICS

Having considered the classical theory of massive electrodynamics with magnetic charge, we can now turn to quantum mechanics. Since 1931, the Dirac quantization condition has been derived in many ways differing by their initial assumptions. Obviously those methods using gauge invariance (such as original Dirac derivation or the Wu-Yang formulation [14]) are not applicable in our case. Other methods [8–11] depend on rotational invariance and we can try to generalize them to include massive electrodynamics, too. Before doing so, let us recall very briefly the essence of the standard arguments.

Consider an electron placed in the field of a magnetic charge g. The angular momentum operator for the electron is given by

$$\mathbf{L} = \mathbf{r} \times (-i\nabla + e\mathbf{A}) + eg\frac{\mathbf{r}}{r} \quad e > 0. \tag{19}$$

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Despite the stange-looking second term, this operator can be shown to obey all the standard requirements of a *bona fide* angular momentum: see commutation relations Eqs. (22–24) below. The angular momentum is the generator of rotations. The two terms are individually not angular momentum operators, in spite of the appearance of the first term, but the sum is an angular momentum operator (see [8] for more details). Moreover, L_i commute with the hamiltonian

$$H = -\frac{1}{2m}(\nabla + ie\mathbf{A})^2 + V(r).$$ (20)

Next, it was shown in [8] that the quantity $\mathbf{L}\frac{\mathbf{r}}{r} = eg$, should be quantized according to

$$eg = 0, \ \pm\frac{1}{2}, \ \pm 1\ldots,$$ (21)

which is the Dirac quantization condition.

Now, we would like to generalize this result to the case of massive electrodynamics. Unfortunately, this turns out to be impossible: we will show that the angular momentum operator cannot be defined for the system of charge plus monopole within massive electrodynamics. More exactly, the following theorem holds.

Theorem

There are no such operators L_i that the following standard properties are satisfied:

$$[L_i, L_j] = i\varepsilon_{ijk}L_k$$ (22)

$$[L_i, r_j] = i\varepsilon_{ijk}r_k$$ (23)

$$[L_i, D_j] = i\varepsilon_{ijk}D_k$$ (24)

where

$$\mathbf{D} = -i\nabla + e\mathbf{A}$$ (25)

is the kinetic momentum operator,

$$\mathbf{D} = m\dot{\mathbf{r}} \tag{26}$$

Note that the conditions of this theorem are not too restrictive: for example, we do *not* require that the hamiltonian be rotationally invariant (i.e., $[L_i, H] = 0$ is not required).

<u>Proof</u>

Our proof consists of two parts. First we prove that from Eqns. (22 − 25) it follows that

$$[L_i, H_j] = i\varepsilon_{ijk}H_k \tag{27}$$

where $\mathbf{H}$ is the magnetic field,

$$\mathbf{H} = \nabla \times \mathbf{A}.$$

Second, we show that Eqn. (27) is incompatible with the general form for $\mathbf{H}$, Eq. (18). Let us start with the first part.

Commuting Eq. (25) with itself, we obtain

$$[D_i, D_j] = -i\varepsilon_{ijk}H_k \tag{28}$$

Therefore, we have this expression for the magnetic field:

$$H_k = \frac{i}{2}\varepsilon_{kij}[D_i, D_j]. \tag{29}$$

We note that this form of the commutator implies non-associativity and the appearance of the three-cocycle [15].

Now, let us plug this expression into the commutator $[L_n, H_k]$:

$$[L_n, H_k] = \left[L_n, \tfrac{i}{2}\varepsilon_{ijk}[D_i, D_j]\right] =$$
$$\tfrac{1}{2}\varepsilon_{kij}(L_nD_iD_j - L_nD_jD_i - D_iD_jL_n + D_jD_iL_n).$$

We then commute L_n and D_j in the two middle terms of the above equation using Eq.(24):

$$(...) = L_nD_iD_j - (i\varepsilon_{njp}D_p + D_jL_n)D_i - D_i(-i\varepsilon_{njp}D_p + L_nD_j) + D_jD_iL_n. \tag{30}$$

The terms containing ε–tensor can be rewritten using Eq. (28):

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$$i\varepsilon_{njp}(D_iD_p - D_pD_i) = -\varepsilon_{njp}\varepsilon_{iqp}H_q = -\delta_{ni}H_j + \delta_{ji}H_n. \tag{31}$$

Next, the rest of Eq. (30) can be transformed to contain only $[LD]$ and $[DD]$ commutators.

$$
\begin{aligned}
L_nD_iD_j - D_jL_nD_i - D_iL_nD_j - D_jD_iL_n &= [L_nD_i, D_j] - [D_iL_n, D_j] \\
&= [[L_n, D_i], D_i] \\
&= [i\varepsilon_{nir}D_r, D_j] \\
&= \varepsilon_{nir}\varepsilon_{rjs}H_s \\
&= \delta_{nj}H_i - \delta_{ij}H_n.
\end{aligned}
\tag{32}
$$

Note that in obtaining Eq. (31) and (32) we have used the identity

$$\varepsilon_{ijk}\varepsilon_{lmk} = \delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}. \tag{33}$$

Finally, putting Eq. (31) and (32) together, we finally obtain Eq. (27). Thus we have shown that indeed Eq. (27) follows from Eqns. (22–25).

Now we turn to the second part of our proof. Let us insert the general form of the magnetic field $\mathbf{H}'$, Eq. (18), into the commutator $[L_i, H_l]$; we shall disregard the $\mathbf{H}^D$ field because it satisfies the correct commutation relations with L due to Eq. (23):

$$
\begin{aligned}
[L_i, H_l'] &= [L_i, hr_l + gn_l] \\
&= -i\varepsilon_{ijk}r_jn_kr_l\frac{\partial h}{\partial(\mathbf{rn})} + i\varepsilon_{ilp}r_p - i\varepsilon_{ijk}r_jn_kn_l\frac{\partial g}{\partial(\mathbf{rn})}.
\end{aligned}
\tag{34}
$$

If L_i were good angular momentum operators, this had to be equal to

$$i\varepsilon_{ilq}H_q' = i\varepsilon_{ilq}(hr_q + gn_q). \tag{35}$$

Now, compare Eqs. (34) and (35). We notice that:

1. terms proportional to h in both equations coincide.

2. terms, quadratic in n_i in Eq. (34) must vanish (because the remaining term in Eq. (35) is linear in n_i and the condition $\mathbf{n}^2 = 1$ has never been used), therefore, we should have

$$\frac{\partial g}{\partial(\mathbf{rn})} = 0.$$

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3. the remaining term (proportional to n_k) in Eq. (34) should be antisymmetric with respect to exchange $i \leftrightarrow l$, in particular, it must vanish for $i = l$. Taking $i = l = 1$; $j = 2$, $k = 3$, we see that this term reduces to

$$-2ixyn_z \frac{\partial h}{\partial(\mathbf{rn})},$$

whence

$$\frac{\partial h}{\partial(\mathbf{rn})} = 0.$$

It follows then that $g = 0$, too.

Thus, we obtain, that the angular momenta with correct commutation relations can exist only if $\mathbf{H}'$ is spherically symmetric:

$$\mathbf{H}' = h(r)\mathbf{r}.$$

But this is impossible, as we noted above, because it is inconsistent with our initial Maxwell equations.

Thus our proof is finished.

IV. CONCLUSIONS

To summarize, we have shown that the introduction of an arbitrary small photon mass makes invalid the existing proofs of the consistency of the Dirac monopole theory. More exactly, the massive electrodynamics does not allow any generalization of the methods in which the Dirac monopole was introduced into the massless electrodynamics. Not only the original Dirac scheme which arrives at the quantization condition by using the gauge invariance, single-valuedness of the wavefunction and "the veto" postulate does not work anymore; but also the different approach relying on the algebra of angular momentum fails in the case of massive electrodynamics. If the magnetic monopole were ever to be introduced into massive electrodynamics consistently, that would be possible only due to some radically new mechanism compared with the existing ones.

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What is the physical reason for that failure?

The whole existence of the Dirac monopole in the massless electrodynamics rests upon the quantization condition which makes invisible the string attached to the monopole. The quantization condition can be obtained either with the help of gauge invariance or the angular momentum quantization. In the massive case, both these approaches are not applicable anymore, as we have shown. That means that there is hardly any way to make the string invisible in the massive electrodynamics.Thus, although the system "string plus monopole" does exist in massive electrodynamics, it is very difficult, if not impossible, to make a consistent theory of the monopole *without* a string.

One may think that our result contradict the principle of continuity which states that any physical consequence of massive electrodynamics should go smoothly into the corresponding result of the standard electrodynamics when the photon mass tends to zero. Indeed, at first sight the appearance of the Dirac monopoles at zero photon mass is an obvious discontinuity as compared with their absence at an arbitrary small photon mass.

However, this simple argument is not yet sufficient to claim discontinuity. An analogy with a similar "discontinuity" is instructive here: consider the number of photon degrees of freedom in massive and massless electrodynamics. the photon with a mass has three polarization states, independent of how small its mass is. Then, as soon as the photon becomes massless, the longitudinal polarization abruptly vanishes and we are left with only two (transverse) polarization states. Does this fact create a discontinuity? No. To find out if there is a discontinuity or not, we have to study the behavior of a more physical quantity, such as the probabilities to emit or absorb a longitudinal photon, rather than merely counting the number of degrees of freedom.

The analysis done by Schrödinger shows that if one considers the interaction of longitudinal photons with matter, this interaction vanishes as photon mass tends to zero. Thus the longitudinal photons decouple in the limit $m_\gamma \to 0$ so that the continuity is restored.

Coming back to our case with monopoles, one should carry out a similar program to make sure the continuity is not violated.

Although we do not intend carrying out this program in the present work, we would like to present here some physical arguments suggesting that the continuity may be indeed preserved. We have shown that instead of a true Dirac monopole, massive electrodynamics contains a more cumbersome object which can be viewed as consisting of three pieces: the monopole, the string and, finally, the additional "diffuse" magnetic field, eq. (17) (or the corresponding vector potential, eq. (14)). Let us call this complex object "a fake monopole". Before any discussion of the continuity problem, one has of course make sure that fake monopoles can be described within a logically consistent quantum theory. This is an open question beyond the present paper, but for the sake of argument let us simply assume that the answer is positive. Then, a natural question arises: how can we physically distinguish between a fake and a true monopoles?

If the fake monopole has a "dequantized" magnetic charge (i.e., the charge not obeying the Dirac quantization condition, eq. (21)) there does not seem to be any problems. However, imagine that a fake monopole takes on a magnetic charge which exactly coincides with one of the quantized values. What happens then? Let us try and detect the string. One way to do that is to use the Aharonov-Bohm effect. But then we run into trouble, because if the magnetic charge has one of the Dirac values, *the strings* of a fake and true monopoles are *exactly the same* . Therefore, *both of them* are *invisible* in the Aharonov-Bohm type of experiment. The way out of this difficulty is to remember about the existence of the feeble "diffuse" component of the fake monopole which is of the order of m^2. Generally speaking, this component would give non-zero contribution to the Aharonov-Bohm effect and thus would, in principle, distinguish between a fake and a true monopoles.

Thus, at least in a gedanken experiment we would be able to distinguish between a fake and a true monopoles. (Whether this gedanken experiment can be transformed into a real one, is an open question.) The observable difference between the fake and true monopoles would vanish *smoothly* as the photon mass tends to zero.

So we see that in this particular case there does not arise any problem of discontinuity. It would be interesting to complement this type of analysis by considering other physi-

cal situations, such as Cabrera-type of experiment or the scattering of the electron on a monopole.

On the other hand, if the continuity is found to be broken (which we cannot completely rule out at this stage), that would constitute a serious argument against the existence of monopoles and thus would provide a possible theoretical reason for the absence of monopoles in nature.

Alternatively, the continuous failure to discover the monopoles in the experiment may be considered as an indirect evidence for the finite photon mass (unfortunately, we cannot say anything definite about the value of this mass).

In this paper we have considered only fundamental monopoles within massive U(1) electrodynamics. A natural question, then, is: what about t'Hooft-Polyakov monopoles which appear necessarily within gauge theories based on simple groups (such as SU(5))? In such theories, one might give small mass to the photon by spontaneous breaking of the electromagnetic U(1) symmetry.

However, there is perhaps less interest in considering such kind of theories because they seem to be ruled out by the experiment. More specifically, the reason is that after giving photon a mass through spontaneous breaking of U(1) symmetry in such theories there arises very light charged particle which is unacceptable experimentally [16,17]. An important feature of such theories is that the electric charge conservation is violated in these theories: to give photon a mass, we need to have a *charged* scalar field with *non-zero* vacuum expectation; this necessarily leads to the electric charge non-conservation.

(To make our our discussion of this point more comprehensive, we note that there does exist an acceptable way to spontaneously violate the electric charge conservation and give photon a mass. But it requires introduction of higgs particles with extremely small electric charge: for details, see [16,18]. However, even this mechanism is extremely difficult or impossible to implement within theories based on semisimple groups (e.g. SU(5)), see the discussion in [19]).

Thus the present work reveals a new and rather general relation between the two fun-

damental facts: the masslessness (massiveness?) of the photon and the non-existence (existence?) of the magnetic monopole.

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FIGURES

FIG. 1. The choice of coordinates for analysing the integral in Eq. (14).